

Diversity Multiplexing Tradeoff of the Half-duplex Slow Fading Multiple Access Channel based on Generalized Quantize-and-Forward Scheme

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Abstract

This paper investigates the Diversity Multiplexing Tradeoff (DMT) of the generalized quantize-and-forward (GQF) relaying scheme over the slow fading half-duplex multiple-access relay channel (HD-MARC). The compress-and-forward (CF) scheme has been shown to achieve the optimal DMT when the channel state information (CSI) of the relay-destination link is available at the relay. However, having the CSI of relay-destination link at relay is not always possible due to the practical considerations of the wireless system. In contrast, in this work, the DMT of the GQF scheme is derived without relay-destination link CSI at the relay. It is shown that even without knowledge of relay-destination CSI, the GQF scheme achieves the same DMT, achievable by CF scheme with full knowledge of CSI.

I. INTRODUCTION

In wireless communication systems, relaying can either increase the system throughput or the reliability by creating a virtual distributed antenna system [1], [2]. In the case of relay cooperating with multiple sources, a Multiple Access Channel with a relay (MARC) has been extensively studied in [3], [4].

Motivated by the practical constraint that delay constraints exist in some wireless channels and relay cannot transmit and receive simultaneously in wireless communications [1], [5], a

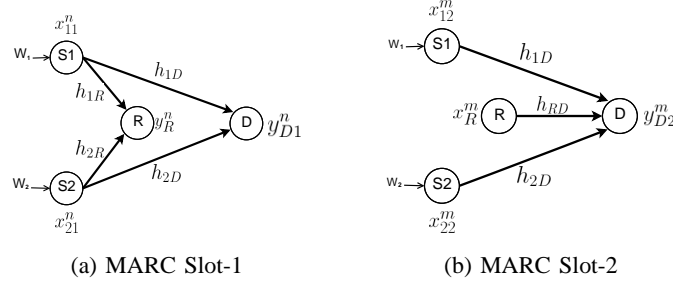


Fig. 1. Message flow of the Half-Duplex Multiple Access Relay Channel.

slow fading Half-Duplex MARC (HD-MARC) (shown in Fig. 1) is considered in this work. In particular, a block fading channel where the channel coefficients stay constant in each block but change independently from block to block is studied. In addition, it is assumed that the channel state information (CSI) is not available at the transmitter side. Specifically, the destination has the receiver CSI and the relay has only the CSI of the source-to-relay link. The performance measure used in this work is the diversity-multiplexing tradeoff (DMT) which was introduced in [6]. The DMT characterizes the multiple-antenna communications in terms of the relationship between system throughput and transmission reliability at asymptotically high signal-to-noise ratio (SNR).

For the HD-MARC, the DMT of different relaying schemes, i.e. dynamic decode-and-forward (DDF), multiaccess amplify-and-forward (MAF) and compress-and-forward (CF), have been characterized in [2], [7], [8]. In [8], it is shown that the CF scheme has its advantages over DDF and MAF scheme in terms of sustaining to multiple antennas case. Besides, the CF scheme is also able to achieve the optimal DMT upper bound when the multiplexing gain is higher than $\frac{4}{5}$. To achieve the optimal DMT, the CF scheme needs to have two assumptions: 1) using Wyner-Ziv coding and 2) the relay has perfect channel state information (CSI). The effect of the Wyner-Ziv coding on the DMT of the CF scheme has been investigated in [9]. In practice, having the CSI of relay-destination link at relay is generally too ideal. When the critical delay constraint exist in the wireless channels, the relay may not be able to obtain the CSI accurately.

In this letter, we investigate the DMT of the HD-MARC based on the GQF scheme. The GQF scheme generalizes the quantize-and-forward (QF) scheme (a variation of the classic CF scheme) to the multiple user channels by taking into account the multiple access at both relay and destination. The QF scheme achieves the optimal DMT for a half-duplex three-node relay channel without the relay-destination link state available at relay [10]. The CF scheme

can achieve the optimal DMT of the symmetric HD-MARC when perfect CSI available at relay and $\frac{4}{5} < r < 1$ [8]. As shown in this work, the DMT achieved by the GQF scheme is

$$d_{GQF}(\bar{r}) = \begin{cases} 2 - r, & \text{if } 0 \leq r \leq \frac{1}{2} \\ 3(1 - r), & \text{if } \frac{1}{2} \leq r \leq 1. \end{cases}$$

With only the source-relay CSI at relay, the GQF scheme achieves the optimal DMT for all the range of multiplexing gain $0 < r < 1$.

II. SYSTEM MODEL AND NOTATIONS

A half-duplex multiple access relay channel is considered in this work (Fig. 1). In particular, two sources S_1 and S_2 wish to communicate with one destination D . Relay helps the information propagation by cooperating with the two sources. Relay operates in the Half-Duplex mode, either receiving signals from the source nodes (S_1 and S_2) or transmitting to the destination D . Assume that each communication block length is totally l channel uses and divided into two slots. The lengths of the first and the second slot are n and m channel uses, respectively. In the first slot, both S_1 and S_2 broadcast their messages to relay and D . In the second slot, S_1 and S_2 keep transmitting to D while relay cooperatively transmitting to D as well. Denote x_{i1}^n and x_{i2}^m , $i \in \{1, 2\}$ as the transmitted sequences by S_i in the first and second slot correspondingly, and x_R^m as the transmitted sequence by the relay node in the second slot, where $x_{ij}^k = [x_{ij,1}, x_{ij,2}, \dots, x_{ij,k}]$ and $x_R^k = [x_{R,1}, x_{R,2}, \dots, x_{R,k}]$. The received sequences at the destination in the first and the second slots are denoted as y_{D1}^n and y_{D2}^m , respectively, and the received sequence at the relay in the first slot is denoted as y_R^n . The channel transition probabilities are described by the following:

$$\begin{aligned} y_{D1}^n &= h_{1D}x_{11}^n + h_{2D}x_{21}^n + z_{D1}^n \\ y_R^n &= h_{1R}x_{11}^n + h_{2R}x_{21}^n + z_R^n \\ y_{D2}^m &= h_{1D}x_{12}^m + h_{2D}x_{22}^m + h_{RD}x_R^m + z_{D2}^m \end{aligned}$$

where h_{ij} for $i \in \{1, 2, R\}$ and $j \in \{R, D\}$ denote the channel coefficients between the transmission node i and the reception node j . For the slow fading channel, these coefficients are random variables and stay constant within each block and changes independently over different blocks. In particularly, a Rayleigh fading model is considered, which means the channel coefficients h_{ij} are assumed to be mutually independent and circularly symmetric complex Gaussian with zero means and variances σ_{ij}^2 . The elements of the noise sequences

of z_{11}^n, z_{12}^m and z_R^n are also circularly symmetric complex Gaussian with zero means and unit variances. For the continuous-valued channels, the transmitters have power constraints over the transmitted sequences as the $\frac{1}{n} \sum_{i=1}^n |x_{j,i}|^2 \leq \text{SNR}$, for $j \in \{11, 21\}$ and $\frac{1}{m} \sum_{i=1}^m |x_{k,i}|^2 \leq \text{SNR}$, for $k \in \{12, 22, R\}$, where $|x|$ shows the absolute value of x .

The following random variables are defined to clarify the input and output relationships of the HD-MARC. Let X_i for $i \in \{11, 21, 12, 22, R\}$, Z_j for $j \in \{D1, D2, R\}$ and Z_Q be generic random variables which are complex Gaussian with zero mean and are mutually independent. The variances of X_i , Z_j and Z_Q are P_i , 1 and σ_Q^2 respectively. The random variable Y_k denotes the channel output where $k \in \{D1, D2, R\}$. The auxiliary random variable \hat{Y}_R is the quantized signal of Y_R , i.e., $\hat{Y}_R = Y_R + Z_Q$ where Z_Q is the quantization noise.

Follows the conventions as in [2], [6], [9], [10], define $f(\text{SNR}) \doteq \text{SNR}^d$ if $\lim_{\text{SNR} \rightarrow \infty} \frac{\log f(\text{SNR})}{\log \text{SNR}} = d$. As DMT discusses the system performance at asymptotically high SNR, we assume all transmitters has the power $P_i = \text{SNR}$. The information rate $R = r \log \text{SNR}$ is increasing with SNR by a fixed ratio r , where $0 < r < 1$. In a slow-fading environment, if the target data rate R is greater than the instantaneous mutual information, then outage event occurs. Denote $P_{out}(R)$ as the outage probability. At high SNR, the outage exponent (diversity gain) is then defined as

$$d(r) = - \lim_{\text{SNR} \rightarrow \infty} \frac{\log P_{out}(r \log \text{SNR})}{\log \text{SNR}},$$

where r is referred as the multiplexing gain. A coding scheme achieves a diversity gain or outage exponent of $d(r)$ for any fixed r when $P_{out}(r \log \text{SNR}) \doteq \text{SNR}^{-d(r)}$.

III. DMT OF THE GQF SCHEME

To derive the DMT of the GQF scheme in the HD-MARC, the achievable rate region and the corresponding outage event and the outage probability is shown first. The DMT result and its discussion is shown in the second part of this section.

A. Achievable Rate Region and Outage Probability

In GQF, relay quantizes its observation Y_R to obtain \hat{Y}_R after the first slot, and then sends the quantization index $u \in \mathcal{U} = \{1, 2, \dots, 2^{LR_U}\}$ in the second slot with X_R . Unlike the conventional CF, no Wyner-Ziv binning is applied. At the destination, decoding is also different in the sense that joint-decoding of the messages from both slots without explicitly decoding the quantization index is performed in GQF scheme. The following theorem shows the achievable rate regions:

Theorem 1: The following rate regions are achievable over discrete memoryless HD-MARC based on the GQF scheme:

$$\begin{aligned} R_1 &< \beta I(X_{11}; Y_{D1}, \hat{Y}_R | X_{21}) \\ &+ (1 - \beta) I(X_{12}; Y_{D2} | X_{22}, X_R) \end{aligned} \quad (1)$$

$$\begin{aligned} R_1 + R_U &< \beta [I(X_{11}, \hat{Y}_R; X_{21}, Y_{D1}) + I(X_{11}; \hat{Y}_R)] \\ &+ (1 - \beta) I(X_{12}, X_R; Y_{D2} | X_{22}) \end{aligned} \quad (2)$$

$$\begin{aligned} R_2 &< \beta I(X_{21}; Y_{D1}, \hat{Y}_R | X_{11}) \\ &+ (1 - \beta) I(X_{22}; Y_{D2} | X_{12}, X_R) \end{aligned} \quad (3)$$

$$\begin{aligned} R_2 + R_U &< \beta [I(X_{21}, \hat{Y}_R; Y_{D1} | X_{11}) + I(X_{21}; \hat{Y}_R)] \\ &+ (1 - \beta) I(X_{22}, X_R; Y_{D2} | X_{12}) \end{aligned} \quad (4)$$

$$\begin{aligned} R_1 + R_2 &< \beta I(X_{11}, X_{21}; Y_{D1}, \hat{Y}_R) \\ &+ (1 - \beta) I(X_{12}, X_{22}; Y_{D2} | X_R) \end{aligned} \quad (5)$$

$$\begin{aligned} R_1 + R_2 + R_U &< \beta [I(X_{11}, X_{21}, \hat{Y}_R; Y_{D1}) + I(X_{11}, X_{21}; \hat{Y}_R)] \\ &+ (1 - \beta) [I(X_{12}, X_{22}, X_R; Y_{D2})], \end{aligned} \quad (6)$$

where $\beta = n/l$ is fixed and

$$R_U > \beta I(Y_R, \hat{Y}_R), \quad (7)$$

for all input distributions

$$p(x_{11})p(x_{21})p(x_{12})p(x_{22})p(x_R)p(\hat{y}_R|y_R).$$

Proof: The detail of the proof is shown in Appendix A. ■

For convenience of derivation, denote the channel coefficient vector in the slow-fading HD-MARC as $\mathbf{h} := [h_{1D}, h_{2D}, h_{1R}, h_{2R}, h_{RD}]$. Given \mathbf{h} , the instantaneous mutual information corresponding to the left hand of (1)-(6) are denoted as $I_{R_i}(\mathbf{h})$, where $i \in \{1, 1u, 2, 2u, 12, 12u\}$.

As the transmitters have no access to the CSI, S_1 and S_2 can only use a fixed rate pair of (R_1, R_2) to send information. The relay node has no CSI of the relay-destination link, therefore it is not able to adapt to the channel state \mathbf{h} but can only assume a fixed rate of R_U for its transmission. In order to do so, the relay selects the auxiliary random variable \hat{Y}_R and chooses the variance of the Z_Q . Since

$$R_U = \beta I(Y_R, \hat{Y}_R) = \beta \log(1 + \frac{1 + |h_{1R}|^2 P_{11} + |h_{2R}|^2 P_{21}}{\sigma_Q^2}) \quad (8)$$

and all the parameters in (8) are known at relay, it can choose any fixed R_U successfully.

In the GQF scheme, the destination node employs the joint-decoding technique, thus the outage event happens when either one of the conditions (1)-(6) not satisfied. The outage event can be defined as the sets of

$$\begin{aligned}
\mathcal{O}_{R_1} &:= \{\mathbf{h} : R_1 > I_{R_1}(\mathbf{h})\} \\
\mathcal{O}_{R_{1u}} &:= \{\mathbf{h} : R_1 + R_U > I_{R_{1u}}(\mathbf{h})\} \\
\mathcal{O}_{R_2} &:= \{\mathbf{h} : R_2 > I_{R_2}(\mathbf{h})\} \\
\mathcal{O}_{R_{2u}} &:= \{\mathbf{h} : R_2 + R_U > I_{R_{2u}}(\mathbf{h})\} \\
\mathcal{O}_{R_{12}} &:= \{\mathbf{h} : R_1 + R_2 > I_{R_{12}}(\mathbf{h})\} \\
\mathcal{O}_{R_{12u}} &:= \{\mathbf{h} : R_1 + R_2 + R_U > I_{R_{12u}}(\mathbf{h})\}
\end{aligned} \tag{9}$$

As in (8), R_U is chosen to satisfy (7), the outage probability of the GQF scheme can be described as

$$P_{out}^{GQF}(R_1, R_2, R_U) = Pr\{\mathcal{O}_{R_1} \cup \mathcal{O}_{R_{1u}} \cup \mathcal{O}_{R_2} \cup \mathcal{O}_{R_{2u}} \cup \mathcal{O}_{R_{12}} \cup \mathcal{O}_{R_{12u}}\}. \tag{10}$$

B. DMT of the GQF scheme

Based on the achievable rates and the outage probability, the DMT of the GQF scheme is derived and discussed in this section.

The DMT upper bound of the symmetric MARC from [2] and [8] is

$$d_{upper}(\bar{r}) = \begin{cases} 2 - r, & \text{if } 0 \leq r \leq \frac{1}{2} \\ 3(1 - r), & \text{if } \frac{1}{2} \leq r \leq 1, \end{cases} \tag{11}$$

when both sources taking the same multiplexing gain of $\frac{r}{2}$, $\bar{r} = (\frac{r}{2}, \frac{r}{2})$. Since the cut-set upper bound results a lower bound on the outage probability, the DMT upper bound of the system can be derived accordingly.

With the previously obtained achievable rates and the outage probability of the GQF scheme, the achievable DMT in the following proposition:

Proposition 1: For the HD-MARC, the GQF scheme achieves the DMT

$$d_{GQF}(\bar{r}) = \begin{cases} 2 - r, & \text{if } 0 \leq r \leq \frac{1}{2} \\ 3(1 - r), & \text{if } \frac{1}{2} \leq r \leq 1. \end{cases} \tag{12}$$

This $d_{GQF}(\bar{r})$ is optimal as it is equal to the upper bound of the DMT of the HD-MARC.

Proof: The detail of the proof is shown in Appendix B. ■

As a reference, the DMT achieved by the CF scheme is shown in the below:

$$d_{CF}(\bar{r}) = \begin{cases} 2(1-r), & \text{if } 0 \leq r \leq \frac{2}{3} \\ 1 - \frac{r}{2}, & \text{if } \frac{2}{3} \leq r \leq \frac{4}{5} \\ 3(1-r), & \text{if } \frac{4}{5} \leq r \leq 1. \end{cases}$$

Based on the range of r , we can compare the DMT results shown above into two different cases.

First, in low multiplexing region $r \leq \frac{1}{2}$, the typical outage event is happened when only one of the sources is in outage. Using time sharing of the relay, the CF scheme can achieve the DMT of 2×1 MISO system, which is the optimum case. However, without time sharing of the relay, the GQF scheme is also able to achieve the optimal DMT. The GQF scheme, similarly as DDF scheme, shows the advantage of DMT.

Second, when $r \geq \frac{1}{2}$, the typical outage event is caused by both of the users are in outage. If $r \geq \frac{4}{5}$, the CF scheme performs better than the DDF scheme [8]. The CF scheme achieves optimal DMT since it compresses both sources together which is more efficient in high data rates. At the same condition, the GQF scheme also achieves the optimal DMT as it is naturally a variation of the classic CF scheme.

In both cases, the CF scheme requires complete CSI available at relay to achieve the optimal DMT in some range of r . However, the GQF achieves the optimal DMT for all ranges of r without having the CSI of relay-destination link at relay.

IV. CONCLUSION

In this paper, the DMT of the GQF scheme has been derived in the slow fading half-duplex MARC. It is shown that the GQF scheme can achieve the optimal DMT when the relay has no access to the CSI of the relay-destination link while the classic CF scheme can only achieve some part of optimal DMT with complete CSI at relay.

APPENDIX A

PROOF OF THEOREM 1

Assume the source messages W_1 and W_2 are independent of each other. Each message W_i , $i \in \{1, 2\}$, is uniformly distributed in its message set $\mathcal{W}_i = [1 : 2^{LR_i}]$.

1) *Codebook Generation:* Assume the joint pmf factors as

$$p(x_{11})p(x_{21})p(x_{12})p(x_{22})p(x_R)p(\hat{y}_R|y_R)p(y_{D1}, y_R|x_{11}, x_{12})p(y_{D2}|x_{12}, x_{22}, x_R). \quad (13)$$

Fix any input distribution $p(x_{11})p(x_{21})p(x_{12})p(x_{22})p(x_R)p(\hat{y}_R|y_R)$, for $k = 1, 2$, randomly and independently generate

- 2^{lR_k} codewords $x_{k1}^n(w_k)$, $w_k \in \mathcal{W}_k$, each according to $\prod_{i=1}^n p_{X_{k1}}(x_{k1,i}(w_k))$;
- 2^{lR_k} codewords $x_{k2}^m(w_k)$, $w_k \in \mathcal{W}_k$, each according to $\prod_{i=1}^m p_{X_{k2}}(x_{k2,i}(w_k))$;
- 2^{lR_U} codewords $x_R^m(u)$, $u \in \mathcal{U} = \{1, 2, \dots, 2^{lR_U}\}$, each according to $\prod_{i=1}^m p_{X_R}(x_{R,i}(u))$.

Calculate the marginal distribution

$$p(\hat{y}_R) = \sum_{x_{11} \in \mathcal{X}, x_{21} \in \mathcal{X}, y_{D1} \in \mathcal{Y}, y_R \in \mathcal{Y}_R} p(\hat{y}_R|y_R)p(y_R, y_{D1}|x_{11}, x_{21})p(x_{11})p(x_{21}),$$

randomly and independently generate 2^{lR_U} codewords $\hat{y}_R^n(u)$, each according to $\prod_{i=1}^n p_{\hat{Y}_R}(\hat{y}_{R,i}(u))$.

2) *Encoding:* To send message w_i , the source node S_i transmits $x_{i1}^n(w_i)$ in the first slot and $x_{i2}^m(w_i)$ in the second slot, where $i \in \{1, 2\}$. Let $\epsilon' \in (0, 1)$. After receiving y_R^n at the end of the first slot, the relay tries to find a unique $u \in \mathcal{U}$ such that

$$(y_R^n, \hat{y}_R^n(u)) \in \mathcal{T}_{\epsilon'}^n(Y_R, \hat{Y}_R) \quad (14)$$

where $\mathcal{T}_{\epsilon'}^n(Y_R, \hat{Y}_R)$ is the ϵ' -strongly typical set as defined in [11]. If there are more than one such u , randomly choose one in \mathcal{U} . The relay then sends $x_R^m(u)$ in the second slot.

3) *Decoding:* The destination D starts decoding the messages after the second slot transmission finishes. Let $\epsilon' < \epsilon < 1$. Upon receiving in both slots, D tries to find a unique pair of the messages $\hat{w}_1 \in \mathcal{W}_1$ and $\hat{w}_2 \in \mathcal{W}_2$ such that

$$(x_{11}^n(\hat{w}_1), x_{21}^n(\hat{w}_2), y_{D1}^n, \hat{y}_R^n(u)) \in \mathcal{T}_{\epsilon'}^n(X_{11}, X_{21}, Y_{D1}, \hat{Y}_R)$$

$$(x_{12}^m(\hat{w}_1), x_{22}^m(\hat{w}_2), x_R^m(u), y_{D2}^m) \in \mathcal{T}_{\epsilon'}^m(X_{12}, X_{22}, X_R, Y_{D2})$$

for some $u \in \mathcal{U}$.

4) *Probability of Error Analysis:* Let W_i denote the message sent from source node S_i , $i \in \{1, 2\}$. U represents the message index chosen by the relay R . Based on the symmetry of the codebook construction and the fact that the messages W_i is chosen uniformly from \mathcal{W}_i , the probability of error averaged on W_i and U over all possible codebooks is

$$Pr(\epsilon) = Pr(\hat{W}_1 \neq 1 \cup \hat{W}_2 \neq 1 | W_1 = 1, W_2 = 1). \quad (15)$$

Define two events \mathcal{E}_0 and $\mathcal{E}_{(w_1, w_2)}$:

$$\mathcal{E}_0 := \{((Y_R^n, \hat{Y}_R^n(u)) \notin \mathcal{T}_{\epsilon'}(Y_R \hat{Y}_R)), \text{ for all } u\} \quad (16)$$

$$\begin{aligned} \mathcal{E}_{(w_1, w_2)} := \{ & (X_{11}^n(w_1), X_{21}^n(w_2), Y_{D1}^n, \hat{Y}_R^n(u)) \in \mathcal{T}_{\epsilon}^n(X_{11} X_{21} Y_{D1} \hat{Y}_R) \text{ and} \\ & (X_{12}^m(w_1), X_{22}^m(w_2), X_R^m(u), Y_{D2}^m) \in \mathcal{T}_{\epsilon}^m(X_{11} X_{21} X_R Y_{D2}) \text{ for some } u\}. \end{aligned} \quad (17)$$

Then $Pr(\epsilon)$ can be rewritten as

$$\begin{aligned} Pr(\epsilon) \leq & Pr(\mathcal{E}_0 | W_1 = 1, W_2 = 1) + Pr((\mathcal{E}_{(1,1)})^c \cap \mathcal{E}_0^c | W_1 = 1, W_2 = 1) \\ & + Pr(\cup_{(w_1, w_2) \in \mathcal{A}} \mathcal{E}_{(w_1, w_2)} | W_1 = 1, W_2 = 1), \end{aligned} \quad (18)$$

where $\mathcal{A} := \{(w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2 : (w_1, w_2) \neq (1, 1)\}$. Assume β is fixed, then by covering lemma [12], $Pr(\mathcal{E}_0 | W_1 = 1, W_2 = 1) \rightarrow 0$ when $l \rightarrow \infty$, if

$$R_U > \beta I(Y_R, \hat{Y}_R) + \delta(\epsilon') \quad (19)$$

where $\delta(\epsilon') \rightarrow 0$ as $\epsilon' \rightarrow 0$. By the conditional typicality lemma [12], $Pr((\mathcal{E}_{(1,1)})^c \cap \mathcal{E}_0^c | W_1 = 1, W_2 = 1) \rightarrow 0$ as $l \rightarrow \infty$. Through some standard probability error analysis [10], the second line of (18), $Pr(\cup_{(w_1, w_2) \in \mathcal{A}} \mathcal{E}_{(w_1, w_2)} | W_1 = 1, W_2 = 1) \rightarrow 0$, for fixed $\beta = \frac{n}{l}$, $1 - \beta = \frac{m}{l}$, if $l \rightarrow \infty$, $\epsilon \rightarrow 0$ and the inequalities (1)-(6) hold. Therefore, the probability of error $P(\epsilon) \rightarrow 0$. The proof completes and the achievable rate region is shown in *Theorem 1*.

APPENDIX B

PROOF OF PROPOSITION 1

The lower bound of the DMT achieved by the GQF scheme will be derived first. Then we show that the lower bound meets the upper bound, hence the optimal DMT is achieved by the GQF scheme. In order to find the lower bound on DMT, we need the following lemma:

Lemma 1: For the case $R_1 = R_2 = \frac{r}{2} \log \text{SNR}$, $R_U = r_U \log \text{SNR}$, and $\beta = r_U = \frac{1}{2}$,

$$\Pr(\mathcal{O}_{R_i}) \doteq \text{SNR}^{-(2-r)} \quad (20)$$

$$\Pr(\mathcal{O}_{R_{12}}) \doteq \text{SNR}^{-4(1-r)} \quad (21)$$

$$\Pr(\mathcal{O}_{R_{12u}}) \doteq \text{SNR}^{-3(1-r)} \quad (22)$$

where \mathcal{O}_{R_i} , $i \in \{1, 1u, 2, 2u\}$, $\mathcal{O}_{R_{12}}$ and $\mathcal{O}_{R_{12u}}$ are the outage events defined previously.

Proof: The detail is shown in the Appendix C. ■

To find a lower bound on the DMT, the union upper bound is applied. The outage probability of the GQF scheme can be upper bounded by

$$\begin{aligned} Pr(\mathcal{O}) &= Pr(\mathcal{O}_{R_1} \cup \mathcal{O}_{R_{1u}} \cup \mathcal{O}_{R_2} \cup \mathcal{O}_{R_{2u}} \cup \mathcal{O}_{R_{12}} \cup \mathcal{O}_{R_{12u}}) \\ &\leq Pr(\mathcal{O}_{R_1}) + Pr(\mathcal{O}_{R_{1u}}) + Pr(\mathcal{O}_{R_2}) + Pr(\mathcal{O}_{R_{2u}}) \\ &\quad + Pr(\mathcal{O}_{R_{12}}) + Pr(\mathcal{O}_{R_{12u}}). \end{aligned} \quad (23)$$

In the symmetric HD-MARC, with any fixed $(\bar{r}) = (\frac{r}{2}, \frac{r}{2})$, β , r_u and $R_1 = R_2 = \frac{r}{2} \log \text{SNR}$, $R_U = r_u \log \text{SNR}$, the outage exponent of the GQF scheme and its lower bound are:

$$\begin{aligned} d_{GQF}(\bar{r}, \beta, r_u) &= - \lim_{\text{SNR} \rightarrow \infty} \frac{\log \Pr(\mathcal{O})}{\log \text{SNR}} \\ &\geq - \lim_{\text{SNR} \rightarrow \infty} \frac{\log \sum_i \Pr(\mathcal{O}_{R_i})}{\log \text{SNR}} \\ &= d_{GQF}^*(\bar{r}, \beta, r_u) \end{aligned} \quad (24)$$

where $i \in \{1, 1u, 2, 2u, 12, 12u\}$ and $d_{GQF}^*(\bar{r}, \beta, r_u)$ denotes the lower bound. Let $d_{R_i}(\bar{r}, \beta, r_u)$ represent the outage exponent achieved by the set \mathcal{O}_{R_i} , we have

$$d_{R_i}(\bar{r}, \beta, r_u) = - \lim_{\text{SNR} \rightarrow \infty} \frac{\log \Pr(\mathcal{O}_{R_i})}{\log \text{SNR}}. \quad (25)$$

When $\text{SNR} \rightarrow \infty$, the union bound outage probability will be dominated by the term with smaller exponent. In other words, the upper bound of the outage probability is mostly determined by the term with smallest diversity order, which is shown as

$$d_{GQF}^*(\bar{r}, \beta, r_u) = \min_{\mathcal{O}_{R_i}} d_{R_i}(\bar{r}, \beta, r_u). \quad (26)$$

For each multiplexing exponent r , the outage exponent can be further optimized with the β and r_u

$$\begin{aligned} d_{GQF}(\bar{r}) &= \max_{\beta, r_u} d_{GQF}(\bar{r}, \beta, r_u) \\ &\geq \max_{\beta, r_u} d_{GQF}^*(\bar{r}, \beta, r_u) \\ &\geq d_{GQF}^*(\bar{r}, \frac{1}{2}, \frac{1}{2}). \end{aligned} \quad (27)$$

From *Lemma 1*, the outage exponents achieved by each of the outage event are $d_{R_1}(\bar{r}, \frac{1}{2}, \frac{1}{2}) = d_{R_{1u}}(\bar{r}, \frac{1}{2}, \frac{1}{2}) = d_{R_2}(\bar{r}, \frac{1}{2}, \frac{1}{2}) = d_{R_{2u}}(\bar{r}, \frac{1}{2}, \frac{1}{2}) = 2-r$, $d_{R_{12}}(\bar{r}, \frac{1}{2}, \frac{1}{2}) = 4(1-r)$ and $d_{R_{12u}}(\bar{r}, \frac{1}{2}, \frac{1}{2}) = 3(1-r)$. $d_{GQF}^*(\bar{r}, \frac{1}{2}, \frac{1}{2})$ is taking the minimum of the above terms, thus

$$d_{GQF}^*(\bar{r}, \frac{1}{2}, \frac{1}{2}) = \begin{cases} 2-r, & \text{if } 0 \leq r \leq \frac{1}{2} \\ 3(1-r), & \text{if } \frac{1}{2} \leq r \leq 1. \end{cases} \quad (28)$$

Notice that when $0 < r < 1$, $3(1 - r)$ is always less than $4(1 - r)$. Therefore, $d_{R_{12u}}(\bar{r}, \frac{1}{2}, \frac{1}{2})$ is smaller than $d_{R_{12}}(\bar{r}, \frac{1}{2}, \frac{1}{2})$ for all values of r .

Since $d_{GQF}^*(\bar{r}, \frac{1}{2}, \frac{1}{2})$ coincides with the upper bound of the symmetric HD-MARC from [2], [8], the GQF scheme achieves the optimal DMT. This finishes the proof of *Proposition 1*.

APPENDIX C

PROOF OF LEMMA 1

Following the similar steps as in [2], [9], [10], let $\alpha_j = -\log|h_j|^2/\log \text{SNR}$ for $j \in \{11, 21, 1R, 2R, RD\}$, $R_1 = R_2 = \frac{r}{2}\log \text{SNR}$, $R_U = r_U \log \text{SNR}$, and $\beta = r_U = \frac{1}{2}$. For $i \in \{1, 1u, 2, 2u, 12, 12u\}$, denote the outage probability

$$Pr(\mathcal{O}_{R_i}) \doteq \text{SNR}^{-d_i}. \quad (29)$$

Then the outage exponent or the diversity order can be derived by [6], [9], [10]

$$d_i = \inf_{\mathcal{O}_{R_i}^+} (\alpha_{11} + \alpha_{1R} + \alpha_{21} + \alpha_{2R} + \alpha_{RD}) \quad (30)$$

where $\mathcal{O}_{R_i}^+$ is the set

$$\mathcal{O}_{R_i}^+ = \{(\alpha_{11}, \alpha_{21}, \alpha_{1R}, \alpha_{2R}, \alpha_{RD}) \in \mathbb{R}^{5+} : \mathcal{O}_{R_i} \text{ occurs}\}. \quad (31)$$

A. outage exponent of d_1

First rewrite \mathcal{O}_{R_1} as

$$\mathcal{O}_{R_1} =: \{R_1 > \beta \log(1 + |h_{11}|^2 P_{11} + \frac{|h_{1R}|^2 P_{11}}{1 + \sigma_Q^2}) + (1 - \beta) \log(1 + |h_{11}|^2 P_{12})\}. \quad (32)$$

Perform the change of variables accordingly, $\mathcal{O}_{R_1}^+$ can be obtained

$$\mathcal{O}_{R_1}^+ = \{(\alpha_{11}, \alpha_{21}, \alpha_{1R}, \alpha_{2R}, \alpha_{RD}) \in \mathbb{R}^{5+} : \frac{r}{2} > \frac{1}{2}(1 - \alpha_{11}, 1 - \alpha_{1R})^+ + \frac{1}{2}(1 - \alpha_{11})^+\} \quad (33)$$

Second, in order to solve the optimization problem of d_1 , the above set can be partitioned into two cases. d_1 takes the minimum of the two solutions.

Case 1: $\alpha_{11} \geq 1$. The inequality in $\mathcal{O}_{R_1}^+$ become

$$r > (1 - \alpha_{11}, 1 - \alpha_{1R})^+. \quad (34)$$

Based on the relationship between α_{11} and α_{1R} , the above can be further divided into:

Case 1.1: $\alpha_{11} \leq \alpha_{1R}$. We have $\alpha_{1R} \geq 1$ and the optimum values of α 's for this case, denoted as a vector α^* , are

$$\alpha^* = (\alpha_{11}^*, \alpha_{21}^*, \alpha_{1R}^*, \alpha_{2R}^*, \alpha_{RD}^*) = (1, 0, 1, 0, 0). \quad (35)$$

Case 1.2: $\alpha_{11} \geq \alpha_{1R}$. When $\alpha_{1R} \geq 1$, then the optimum α^* are the same as (35). However, if $\alpha_{1R} \leq 1$, then (34) become

$$r > 1 - \alpha_{1R}. \quad (36)$$

The optimum α^* is then

$$\alpha^* = (1, 0, (1 - r)^+, 0, 0). \quad (37)$$

Let d_{1-1} denote the minimum of the outage exponent in Case 1. Combining Case 1.1 and Case 1.2 gives

$$d_{1-1} = 2 - r \quad (38)$$

Case 2: $\alpha_{11} \leq 1$. The inequality in $\mathcal{O}_{R_1}^+$ changes to

$$r > (1 - \alpha_{11}, 1 - \alpha_{1R})^+ + (1 - \alpha_{11}). \quad (39)$$

Similarly as Case 1, Case 2 is also divided into two cases.

Case 2.1: $\alpha_{11} \leq \alpha_{1R}$. Then (39) becomes

$$r > (1 - \alpha_{11}) + (1 - \alpha_{11}). \quad (40)$$

This leads the optimum α^* to be $(1 - \frac{r}{2}, 0, 1 - \frac{r}{2}, 0, 0)$.

Case 2.2: $\alpha_{11} \geq \alpha_{1R}$. (39) changes to

$$r > (1 - \alpha_{1R}) + (1 - \alpha_{11}). \quad (41)$$

This implies

$$\alpha_{11} + \alpha_{1R} > 2 - r. \quad (42)$$

Choosing α_{21}, α_{2R} and α_{RD} equal to zero, the minimum of the outage exponent for Case 2.2 is $2 - r$. Combining Case 2.1 and Case 2.2, we have

$$d_{1-2} = 2 - r, \quad (43)$$

where d_{1-2} denotes the minimum of the outage exponent in Case 2.

In the last, combining Case 1 and Case 2 and given $d_1 = \min(d_{1-1}, d_{1-2})$ we conclude

$$d_1 = 2 - r = 2(1 - \frac{r}{2}). \quad (44)$$

B. outage exponent of d_{1u} , d_2 and d_{2u}

Similarly as deriving d_1 , rewrite $\mathcal{O}_{R_{1u}}$ as

$$\begin{aligned} \mathcal{O}_{R_{1u}} =: \{R_1 + R_u > \beta \log[(1 + |h_{11}|^2 P_{11}) (\frac{1 + \sigma_Q^2 + |h_{1R}|^2 P_{11} + |h_{2R}|^2 P_{21}}{1 + \sigma_Q^2})] \\ + (1 - \beta) \log(1 + |h_{11}|^2 P_{12} + |h_{RD}|^2 P_R)\}. \end{aligned} \quad (45)$$

Then we have $\mathcal{O}_{R_{1u}}^+$ as

$$\begin{aligned} \mathcal{O}_{R_{1u}}^+ = \{(\alpha_{11}, \alpha_{21}, \alpha_{1R}, \alpha_{2R}, \alpha_{RD}) \in \mathbb{R}^{5+} : \\ r + 1 > (1 - \alpha_{11})^+ + (1 - \alpha_{1R}, 1 - \alpha_{2R})^+ + (1 - \alpha_{11}, 1 - \alpha_{RD})^+\}. \end{aligned} \quad (46)$$

Next, we solve the optimization problem of d_{1u} . Notice that in $\mathcal{O}_{R_{1u}}^+$, $(1 - \alpha_{1R}, 1 - \alpha_{2R})^+$ has three possible outcomes $1 - \alpha_{1R}$, $1 - \alpha_{2R}$ and 0. Each of $(1 - \alpha_{11})^+$ and $(1 - \alpha_{11}, 1 - \alpha_{RD})^+$ has two possible outcomes. Based on these outcomes, $\mathcal{O}_{R_{1u}}^+$ can be partitioned into twelve cases. The derivation of the outage exponent for each of these cases is similar to previous subsection. The result of these cases are shown in the Table I.

The eventual outage exponent of d_{1u} takes the smallest value from the last column of the Table I. Therefore, we have

$$d_{1u} = 2 - r = 2(1 - \frac{r}{2}). \quad (47)$$

Similarly as d_1 and d_{1u} , we can find d_2 and d_{2u} as

$$d_2 = d_{2u} = 2 - r = 2(1 - \frac{r}{2}). \quad (48)$$

C. outage exponent of d_{12} and d_{12u}

Following the similar process as previous subsections, we may rewrite $\mathcal{O}_{R_{12}}$ and $\mathcal{O}_{R_{12u}}$ as

$$\begin{aligned} \mathcal{O}_{R_{12}} =: \{R_1 + R_2 > \beta \log(1 + |h_{11}|^2 P_{11} + |h_{21}|^2 P_{21} \\ + \frac{(h_{11}h_{2R} - h_{1R}h_{21})^2 P_{11}P_{21} + |h_{1R}|^2 P_{11} + |h_{2R}|^2 P_{21}}{1 + \sigma_Q^2}) \\ + (1 - \beta) \log(1 + |h_{11}|^2 P_{12} + |h_{21}|^2 P_{22})\}. \end{aligned} \quad (49)$$

$$\begin{aligned} \mathcal{O}_{R_{12u}} =: \{R_1 + R_2 + R_u > \beta \log(\frac{(1 + |h_{11}|^2 P_{11} + |h_{21}|^2 P_{21})(1 + \sigma_Q^2 + |h_{1R}|^2 P_{11} + |h_{2R}|^2 P_{21})}{1 + \sigma_Q^2}) \\ + (1 - \beta) \log(1 + |h_{11}|^2 P_{12} + |h_{21}|^2 P_{22} + |h_{RD}|^2 P_R)\}. \end{aligned} \quad (50)$$

TABLE I
DIFFERENT CASES OF OPTIMIZATION FOR d_{1u}

Case No.	Outcomes from			Minimum outage exponent
	$(1 - \alpha_{1R}, 1 - \alpha_{2R})^+$	$(1 - \alpha_{11})^+$	$(1 - \alpha_{11}, 1 - \alpha_{RD})^+$	
Case 1-1-1	$1 - \alpha_{1R}$	0	$1 - \alpha_{RD}$	$2 - r$
Case 1-1-2	$1 - \alpha_{1R}$	0	0	$2 - r$
Case 1-2-1	$1 - \alpha_{1R}$	$1 - \alpha_{11}$	$1 - \alpha_{11}$	$2 - r$
Case 1-2-2	$1 - \alpha_{1R}$	$1 - \alpha_{11}$	$1 - \alpha_{RD}$	$2 - r$
Case 2-1-1	$1 - \alpha_{2R}$	0	$1 - \alpha_{RD}$	$2 - r$
Case 2-1-2	$1 - \alpha_{2R}$	0	0	2
Case 2-1-1	$1 - \alpha_{2R}$	$1 - \alpha_{11}$	$1 - \alpha_{11}$	$2 - r$
Case 2-2-2	$1 - \alpha_{2R}$	$1 - \alpha_{11}$	$1 - \alpha_{RD}$	$2 - r$
Case 3-1-1	0	0	$1 - \alpha_{RD}$	3
Case 3-1-2	0	0	0	4
Case 3-2-1	0	$1 - \alpha_{11}$	$1 - \alpha_{RD}$	$3 - r$
Case 3-2-2	0	$1 - \alpha_{11}$	$1 - \alpha_{11}$	$3 - r$

Then the corresponding $\mathcal{O}_{R_{12}}^+$ and $\mathcal{O}_{R_{12u}}^+$ are

$$\mathcal{O}_{R_{12}}^+ = \{(\alpha_{11}, \alpha_{21}, \alpha_{1R}, \alpha_{2R}, \alpha_{RD}) \in \mathbb{R}^{5+} : \\ 2r > (1 - \alpha_{11}, 1 - \alpha_{21}, 1 - \alpha_{1R}, 1 - \alpha_{2R})^+ + (1 - \alpha_{11}, 1 - \alpha_{21})^+\}. \quad (51)$$

$$\mathcal{O}_{R_{12u}}^+ = \{(\alpha_{11}, \alpha_{21}, \alpha_{1R}, \alpha_{2R}, \alpha_{RD}) \in \mathbb{R}^{5+} : \\ 2r + 1 > (1 - \alpha_{11}, 1 - \alpha_{21})^+ + (1 - \alpha_{1R}, 1 - \alpha_{2R})^+ + (1 - \alpha_{11}, 1 - \alpha_{21}, 1 - \alpha_{RD})^+\} \quad (52)$$

Next, we solve the optimization problem of d_{12} and d_{12u} . $\mathcal{O}_{R_{12}}^+$ and $\mathcal{O}_{R_{12u}}^+$ are partitioned into fifteen and thirty six cases respectively. The outage exponent results are shown in Table II and Table III.

The eventual outage exponent of d_{12} and d_{12u} takes the smallest value from the last column of the Table II and Table III . Therefore, we have

$$d_{12} = 4 - 4r = 4(1 - r) \quad (53)$$

$$d_{12u} = 3 - 3r = 3(1 - r). \quad (54)$$

TABLE II
DIFFERENT CASES OF OPTIMIZATION FOR d_{12}

Case No.	Outcomes from		Minimum outage exponent
	$(1 - \alpha_{11}, 1 - \alpha_{21})^+$	$(1 - \alpha_{11}, 1 - \alpha_{21}, 1 - \alpha_{1R}, 1 - \alpha_{2R})^+$	
Case 1-1	$1 - \alpha_{11}$	$1 - \alpha_{11}$	$4(1 - r)$
Case 1-2	$1 - \alpha_{11}$	$1 - \alpha_{21}$	$4(1 - r)$
Case 1-3	$1 - \alpha_{11}$	$1 - \alpha_{1R}$	$4(1 - r)$
Case 1-4	$1 - \alpha_{11}$	$1 - \alpha_{2R}$	$4(1 - r)$
Case 1-5	$1 - \alpha_{11}$	0	4
Case 2	Similar to Case 1		..
Case 3-1	0	$1 - \alpha_{11}$	4
Case 3-2	0	$1 - \alpha_{21}$	4
Case 3-3	0	$1 - \alpha_{1R}$	$4(1 - r)$
Case 3-4	0	$1 - \alpha_{2R}$	$4(1 - r)$
Case 3-5	0	0	4

The proof for Lemma 1 is finished as we find $d_1 = d_{1u} = d_2 = d_{2u} = 2(1 - \frac{r}{2})$, $d_{12} = 4(1 - r)$ and $d_{12u} = 3(1 - r)$.

REFERENCES

- [1] J. Laneman and G. Wornell, "Distributed space-time-coded protocols for exploiting cooperative diversity in wireless networks," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2415 – 2425, oct. 2003.
- [2] K. Azarian, H. El-Gamal, and P. Schniter, "On the achievable diversity-multiplexing tradeoff in half-duplex cooperative channels," *Information Theory, IEEE Transactions on*, vol. 51, no. 12, pp. 4152–4172, 2005.
- [3] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *Inf. Theory, IEEE Trans*, vol. 51, no. 9, pp. 3037 – 3063, sep 2005.
- [4] D. Gunduz, O. Simeone, A. Goldsmith, H. Poor, and S. Shamai, "Multiple multicasts with the help of a relay," *Inf. Theory, IEEE Trans*, vol. 56, no. 12, pp. 6142 –6158, dec. 2010.
- [5] M. Khojastepour, A. Sabharwal, and B. Aazhang, "On capacity of gaussian 'cheap' relay channel," in *Global Telecommunications Conference, 2003. GLOBECOM '03. IEEE*, vol. 3, dec. 2003, pp. 1776 – 1780 vol.3.
- [6] L. Zheng and D. Tse, "Diversity and multiplexing: a fundamental tradeoff in multiple-antenna channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1073–1096, 2003.
- [7] D. Chen, K. Azarian, and J. Laneman, "A case for amplify-forward relaying in the block-fading multiple-access channel," *Information Theory, IEEE Transactions on*, vol. 54, no. 8, pp. 3728–3733, 2008.
- [8] M. Yuksel and E. Erkip, "Multiple-antenna cooperative wireless systems: A diversity-multiplexing tradeoff perspective," *Information Theory, IEEE Transactions on*, vol. 53, no. 10, pp. 3371–3393, 2007.

TABLE III
DIFFERENT CASES OF OPTIMIZATION FOR d_{12u}

Case No.	Outcomes from			Minimum outage exponent
	$(1 - \alpha_{11}, 1 - \alpha_{21})^+$	$(1 - \alpha_{1R}, 1 - \alpha_{2R})^+$	$(1 - \alpha_{11}, 1 - \alpha_{21}, 1 - \alpha_{RD})^+$	
Case 1-1-1	$1 - \alpha_{11}$	$1 - \alpha_{1R}$	$1 - \alpha_{11}$	$3 - 3r$
Case 1-1-2	$1 - \alpha_{11}$	$1 - \alpha_{1R}$	$1 - \alpha_{21}$	$3 - 3r$
Case 1-1-3	$1 - \alpha_{11}$	$1 - \alpha_{1R}$	$1 - \alpha_{RD}$	$3 - 3r$
Case 1-1-4	$1 - \alpha_{11}$	$1 - \alpha_{1R}$	0	3
Case 1-2	Similar to Case 1-1			..
Case 1-3-1	$1 - \alpha_{11}$	0	$1 - \alpha_{11}$	$3.5 - 3r$
Case 1-3-2	$1 - \alpha_{11}$	0	$1 - \alpha_{21}$	$3.5 - 3r$
Case 1-3-3	$1 - \alpha_{11}$	0	$1 - \alpha_{RD}$	$3 - 2r$
Case 1-3-4	$1 - \alpha_{11}$	0	0	5
Case 2	Similar to Case 1			..
Case 3-1-1	0	$1 - \alpha_{1R}$	$1 - \alpha_{11}$	3
Case 3-1-2	0	$1 - \alpha_{1R}$	$1 - \alpha_{21}$	3
Case 3-1-3	0	$1 - \alpha_{1R}$	$1 - \alpha_{RD}$	$3 - 2r$
Case 3-1-4	0	$1 - \alpha_{1R}$	0	3
Case 3-2	Similar to Case 3-1			..
Case 3-3-1	0	0	$1 - \alpha_{11}$	5
Case 3-3-2	0	0	$1 - \alpha_{21}$	5
Case 3-3-3	0	0	$1 - \alpha_{RD}$	4
Case 3-3-4	0	0	0	5

- [9] T. Kim, M. Skoglund, and G. Caire, “Quantifying the loss of compress-forward relaying without wyner-ziv coding,” *Information Theory, IEEE Transactions on*, vol. 55, no. 4, pp. 1529–1533, 2009.
- [10] S. Yao, T. Kim, M. Skoglund, and H. Poor, “Half-duplex relaying over slow fading channels based on quantize-and-forward,” *IEEE Trans. Inf. Theory*, vol. 59, no. 2, pp. 860–872, 2013.
- [11] S. Lim, Y.-H. Kim, A. El Gamal, and S.-Y. Chung, “Noisy network coding,” *IEEE Trans. Inf. Theory*, vol. 57, no. 5, pp. 3132–3152, 2011.
- [12] A. E. Gamal and Y.-H. Kim, “Lecture notes on network information theory,” *CoRR*, vol. abs/1001.3404, 2010.